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TEICHMÜLLER INEQUALITIES WITHOUT COEFFICIENT NORMALIZATION

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Abstract. Teichmüller's relation between the coefficients of extremal schlicht functions and quadratic differentials is extended. The coefficient normalization hypothesis in his theorem is dropped with the result that the new coefficient relations become more complex. This completes the partial result in this direction which is contained in Jenkins' General Coefficient Theorem. A modification of the version of the length-area method used by Teichmüller and Jenkins is introduced in our proof.

1. **Introduction.** In the theory of schlicht functions there is a rather vague but heuristic statement known as *Teichmüller's Principle* [6]. The problem is to give precise formulations of it. The original explicit inequalities of Teichmüller are either contained in or derivable from the much more extensive formulation of this principle, Jenkins' *General Coefficient Theorem* [2]. In its present form precise formulations are given when (1) $d\zeta^2$ is a positive (meromorphic) quadratic differential with poles on a finite Riemann surface R and (2) h is an "admissible" analytic homeomorphism which maps its "admissible" open set $D \subset R$ into R. An admissible function h leaves poles of $d\zeta^2$ in D fixed and omits poles not in D from its range. In addition, at a pole of order greater than 3 there is a rather severe coefficient normalization on h.

This troublesome coefficient normalization is perhaps the most obvious obstruction to application of this principle to high order coefficient problems such as the resolution of Bieberbach's Conjecture. In this regard, we have only been able to use the present form to prove in general that $\operatorname{Re} A_n \leqq n$, when A_k is real for $k \leqq [n/2]$, [4].

The main purpose of this paper is to present a method which can be used to formulate Teichmüller's Principle even in the complete absence of coefficient normalization. What this amounts to is a distillation and refinement of the length-area method used by Jenkins. The approach in this paper can also be applied in

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more general situations than mentioned above. For example we may use this method to consider problems where the range of h is in a surface S other than R. However, to avoid the inherent complications of a more general situation, we shall restrict ourselves in this paper to one of the more classical cases considered by Teichmüller.

Although our results are extensions of Teichmüller's inequality, they differ in character from their predecessors in two ways. First, in each situation there is a parametrized family of sharp inequalities rather than only one. There is also a wider but entirely analogous class of extremal functions. Second, in the non-normalized case our coefficient relations are in general hyperelliptic in essence rather than algebraic as before.

With the exception of the general theory of geodesics of quadratic differentials on finite Riemann surfaces, this paper is intended to be elementary and self-contained. The facts of this theory pertinent to our case are reviewed in §2, the reader being referred to Jenkins [1, Chapter III], [3] for the proofs. Our format is to present the preparatory facts in §§3, 4. The core of the paper is then §5 where we describe the procedure in terms of Jenkins' development of the length-area method and state our main results. The proof and subsequent remarks are contained in the remaining sections.

2. **Quadratic differentials.** In this section we recall the well-known facts and terminology of quadratic differentials on finite Riemann surfaces, which will be used in this paper. Our standard reference is Jenkins' work [1, Chapter III], [3].

(a) Let $\{\pi=(z, U)\}$ denote a defining system of parameters $z\colon U \to C$ for a Riemann surface R. A quadratic differential $d\zeta^2$ assigns a meromorphic function $Q_\pi(z)$ on $z(U)$ to each parameter π, in such a way that the relation

$$Q_\sigma(z) = Q_\tau(w(z))(w'(z))^2$$

holds on $z(U \cap V)$ for any pair of parameters $\sigma=(z, U)$, $\tau=(w, V)$. Here $w=w(z)$ denotes the conformal connecting map $w \circ z^{-1}\colon z(U \cap V) \to w(U \cap V)$, defined by the change in parameters on R. In view of this relation all information about $d\zeta^2$ is contained in $Q_\pi(z)$. Hence the notation $d\zeta^2=Q_\pi(z)\,dz^2$ can be used where $Q_\pi(z)$ is given explicitly.

If $z=z(p)$ is a parameter at $p_0 \in R$ for which $z(p_0)=z_0$, then, near z_0, $Q_\pi(z)$ has the following Laurent expansion:

$$Q_\pi(z) = \sum_{n=e}^{\infty} \alpha_n (z-z_0)^n \qquad (\alpha_e \neq 0). \tag{2.1}$$

Although the coefficients α_n depend on the choice of π the integer $e=e(p; d\zeta^2)$ does not. e is called the *index* of $d\zeta^2$ at p; $|e|$ is called the *order* of $d\zeta^2$ at p. The following points and point sets may be defined in terms of the index e:

$Z = \{e > 0\}$,	the *zeros*,
$P = \{e < 0\}$,	the *poles*,
$A = \{e = 0\}$,	the *regular* points,
$C = \{e \neq 0\}$,	the *critical* points,
$H = \{e \leqq -2\}$,	the *infinite* points,
$F = \{e \geqq -1\}$,	the *finite* points,
$G = \{e \geqq -1, e \neq 0\}$,	the *finite critical* points.

(b) The *Q-metric* $|d\zeta|$ and the *Q-density* $|d\zeta|^2$ are defined on F in terms of the parameter π by the equations $|d\zeta| = |Q_\pi(z)|^{1/2}|dz|$, $|d\zeta|^2 = |Q_\pi(z)|\, dx\, dy$. The restriction of $|d\zeta|$ to A is a conformal metric. If $F = A \cup Z$, in other words if $d\zeta^2$ has no simple poles, then $|d\zeta|$ is a complete Riemannian metric. This occurs in the case of interest of this paper. In the general case a *Q-geodesic* is an arc γ with the "shortest join" property. At each point of γ there is a neighborhood N in which every subarc of γ in N is the shortest join of its end points among all arcs joining these points.

An *isometry* is a homeomorphism h from a domain $D \subset F(d\zeta^2)$ into $F(d\omega^2)$ which preserves distance. Evidently h also preserves geodesics. If $d\omega^2 = d\zeta^2$ then h is an *autometry*. The Q-integrals $\zeta = \int [Q(z)]^{1/2}\, dz$, on domains where they are conformal, are canonical examples of isometries into $F(dz^2)$, the complex plane. They are used to give a standard system of direction. For example, a *horizontal* (*vertical*) geodesic is a Jordan arc or curve which is mapped by ζ into a horizontal (vertical) line. A *trajectory* (*orthogonal trajectory*) is a maximal critical point free horizontal (vertical) geodesic. We say a trajectory γ is *unobstructed* if (1) γ is a Jordan curve or (2) ζ maps γ onto the real line. A *horizontal* isometry preserves horizontal geodesics. (Note that h is a horizontal isometry from $D \subset F(d\zeta^2)$ into $F(d\omega^2)$ if and only if $\omega \circ h \circ \zeta^{-1} \equiv \zeta + d$ locally on $\zeta[A(d\zeta^2) \cap h^{-1}(A(d\omega^2))]$.) Finally, a *translation* is a horizontal autometry and a *translation along trajectories* sends subarcs of a trajectory T in D into T itself.

(c) Given a piecewise smooth Jordan arc $\gamma = \gamma(t)$ in $F(d\zeta^2)$, we may continue $\zeta(z)$ analytically along both sides of γ because $\zeta(z)$ is analytic inside and continuous on the closure of $U(t)$, a half neighborhood of $\gamma(t)$ for each t. Let us denote the *sides* of γ (prime ends of γ with respect to $R \backslash \gamma$ with the appropriate topology) by γ_r, γ_l. If ζ_r, ζ_l are the analytic continuations of ζ along γ_r, γ_l then, since $\zeta_j(\gamma(t \pm \varepsilon)) \neq \zeta_j(\gamma(t))$ for ε small, we may define the *right* or *left* ζ-*angle* at $\gamma(t)$ by the formula

$$\lim_{\varepsilon \downarrow 0} \arg \left[\frac{\zeta_j(\gamma(t+\varepsilon)) - \zeta_j(\gamma(t))}{\zeta_j(\gamma(t-\varepsilon)) - \zeta_j(\gamma(t))} \right] = \psi_j(t), \tag{2.2}$$

for $j = r$ or l and arg, a continuous branch of the argument function. It is useful to describe geodesics in terms of the ζ-angles.

LEMMA (2.3). *A piecewise smooth Jordan arc $\gamma=\gamma(t)$ in F is a geodesic of $d\zeta^2$ if and only if every ζ-angle $\psi_j(t)$ on both sides of γ satisfies $\psi_j(t)\geqq\pi$.*

(2.4) *A simple pole of $d\zeta^2$ ($e=-1$) is never the interior point of a geodesic.*

(2.5) *If p is an interior point of a geodesic γ of index $e\geqq 0$ then the two ζ-angles at p are θ and $\pi(e+2)-\theta$. Hence $\pi\leqq\theta\leqq(e+1)\pi$, and if $e=0$, $\theta=\pi$.*

(2.6) *If* arg *is a continuous branch of the argument function, $\gamma=\gamma(t)$ is a geodesic, and $\gamma(t)\in A$ for $t_0<t<t_1$, then* $\arg[\zeta(\gamma(t))-\zeta(\gamma(t_0))]$ *is constant for $t_0<t<t_1$.*

These remarks are similar to remarks in Jenkins' book [1, §3.2]. They follow from a study of the local behavior of ζ.

(d) *Trajectory structure* on finite Riemann surfaces is defined in terms of regular curve families of horizontal geodesics. *Local* trajectory structure at p is characterized by the index $e(p)$. *Global* trajectory structure is given in terms of a finite number of disjoint canonical domains which we call *Q-domains.* Let K be a component of $\{p\in T : T \text{ is unobstructed}\}$. The interior of the closure of K is a Q-domain.

In three exceptional cases the Q-domain is the whole surface R, which is either a *sphere* or a *torus.* There are five other types of Q-domains called *end, strip, circle, ring* and *density* domains. We recall the following facts concerning Q-domains for later use. For proofs, see [1, §§3.3, 3.4].

LEMMA (2.7). *End domains E are bounded by a curve γ containing one pole p_0 of order $|e|\geqq 3$ and at least one zero of $d\zeta^2$. The ζ-integral maps E conformally onto $\{\pm\operatorname{Im}\zeta>c\}$. At p_0 there are $(|e|-2)$ end domains E_k corresponding to the $(|e|-2)$ vertical directions $\theta_k+\pi/(|e|-2)$ where the horizontal directions*

$$\theta_k = -[\arg\alpha_e+2\pi(k-1)]/(|e|-2) \qquad (1 \leqq k \leqq |e|-2)$$

are defined by the constant α_e of formula (2.1). *The ends of the unobstructed trajectories, which sweep out E_k, tend to p_0 in the asymptotic directions θ_k, θ_l respectively, where $l\equiv(k+1)\ (\mathrm{mod}\ (|e|-2))$.*

LEMMA (2.8). *Strip domains S are bounded by two Jordan arcs γ_1,γ_2 each of which contains at least one finite critical point. The integral ζ maps S conformally onto a strip $\{|\operatorname{Im}(\zeta-\zeta_0)|<w/2\}$. The constant w is called the "width" of S. The unobstructed trajectories, which sweep out S, tend to an infinite point p_1 in the direction $\theta(p_1)$ at the one end, to an infinite point p_2 in the direction $\theta(p_2)$ at the other end.*

LEMMA (2.9). *The circle, ring and density domains are Jordan domains, whose closures contain closed geodesics of finite $|d\zeta|$-length.*

LEMMA (2.10). *Boundary arcs of the Q-domains have ζ-angles π at every finite point on the side facing the Q-domain it bounds.*

3. **Teichmüller's case.** (a) Teichmüller formulated his principle explicitly in several cases. We shall isolate one case and formulate the problem precisely there.

Henceforth, the situation ($d\zeta^2$, R; h, D) described below will be called *Teichmüller's case*.

Let m be a positive integer. On R the Riemann sphere, let $d\zeta^2 = Q(z)\,dz^2$ be a quadratic differential which has the following canonical form:

$$Q(z) = \alpha_{m-1}z^{m-1}\left(1+\sum_{1}^{m-1}\beta_v z^{-v}\right), \tag{3.1}$$

with $\alpha_{m-1}\neq 0$. Notice that $d\zeta^2$ is analytic on R apart from a pole of order $(m+3)$ at ∞. An *admissible* domain D with respect to $d\zeta^2$ is the complement in R of a finite number of horizontal geodesics of finite length. (It will follow that in this case D contains ∞ and is connected.) Living on this domain is the classical family $\Sigma(D)$ of schlicht functions h which have the following normalized Laurent expansion at ∞:

$$h(z) = z+\sum_{n=1}^{\infty} c_n z^{-n}. \tag{3.2}$$

We may now state *Teichmüller's Inequality.*

THEOREM (TEICHMÜLLER). *In Teichmüller's case if h has the coefficient normalization*

$$c_n = 0 \quad \textit{for } 1 \leqq n \leqq m-1, \tag{3.3}$$

then

$$\operatorname{Re}\{\alpha_{m-1}c_m\} \leqq 0. \tag{3.4}$$

Equality occurs in (3.4) *if and only if* $f(z)\equiv z$.

(b) Our problem is to allow all schlicht functions on D rather than those with condition (3.3). The inequalities (3.4) and the associated extremal functions, at which equality is attained, become more complicated. As a first step we note that a change of variables by a linear transformation will place the nonnormalized problem in Teichmüller's case. Second, we note that the General Coefficient Theorem [2] allows the relaxation of (3.3) to the weaker conditions:

$$c_n = 0 \quad \text{for } 1 \leqq n \leqq [(m-2)/2], \tag{3.5}$$

where $[x]$ denotes the greatest integer in x. In this case, it is the translations of D along trajectories, which are the extremal functions. Henceforth, (3.5) will be called *Jenkins' normalization.*

We should mention that Teichmüller's case is very closely related to the case studied by Schaeffer and Spencer in their book [5]. The only real difference is that their quadratic differentials may admit one simple pole. In that occurrence the trick used there of inducing a situation on a two sheeted covering surface of the sphere, branched at the two poles, will lead to Teichmüller's case with a pole of even order at ∞. In fact the induced quadratic differential $4z^2Q(z^2)\,dz^2$ is even.

(c) Certain properties are characteristic of Teichmüller's case. We shall review and develop the properties which will be used in our proof. We are especially concerned with the properties related to the asymptotic directions of trajectories at ∞. The first property concerns the uniqueness of trajectories in this case.

LEMMA (3.6). *In Teichmüller's case, any two points in* $F=A\cup Z=R\backslash\{\infty\}$ *are joined by a unique geodesic* $g(z_1, z_2)$.

Lemma (3.6) is Lemma XXI in [5]. It is also an immediate specialization of Jenkins' Lemma (4.4) in [1, p. 54] (since there is only 1 pole at ∞ in Teichmüller's case, the complement is simply connected).

COROLLARY (3.7). *Admissible domains in Teichmüller's case are connected.*

This follows from the Jordan curve theorem because (3.6) implies the absence of closed geodesics in this case. Referring to (2.9) we see that this fact also implies a simplified global trajectory structure, namely

COROLLARY (3.8). *The Q-domains in Teichmüller's case consist of end and strip domains alone.*

It is necessary that we examine the trajectory structure in Teichmüller's case in detail, paying close attention to the $m+1$ *asymptotic directions* of trajectories at ∞. Specifically these are

$$\theta_k = [-\arg(\alpha_{m-1})+2\pi(k-1)]/(m+1), \tag{3.9}$$

for $1\leqq k\leqq m+1$ where $\alpha_{m-1}\neq 0$ is the leading coefficient in form (3.1) for $d\zeta^2$.

LEMMA (3.10). *In Teichmüller's case there are precisely* $(m+1)$ *end domains* E_k *and at most* $m-2$ *strip domains* S_j. *Unobstructed trajectories in* E_k *emanate from* ∞ *at angle* θ_k *and terminate at* ∞ *at angle* θ_l, *where* $l\equiv(k+1) \pmod{(m+1)}$; *in* S_j *they emanate from* ∞ *at angle* $\theta_{k(j)}$ *and terminate at* ∞ *at angle* $\theta_{l(j)}$, *where* $k(j)+2\leqq l(j) \bmod (m+1)$. *The direction pairs* $\{(\theta_k, \theta_l)\}$ *of the Q-domains are distinct and do not separate one another.*

Proof. We know from the general theory the asymptotic directions of trajectories at ∞ and the structure of the end domains $\{E_k\}_1^{m+1}$ there. We also know that the trajectories in one end of a strip domain tend to ∞ in the same direction.

Let (θ_k, θ_l) be the directions of an unobstructed trajectory T. The uniqueness condition (3.6) implies $l\not\equiv k \pmod{(m+1)}$, for in that event two points on the ends could also be joined by an orthogonal trajectory.

The disjointness of Q-domains and the Jordan curve theorem imply that pairs do not separate one another (i.e. $k_1<k_2<l_1<l_2$). This argument also shows that if T_1, T_2 have the same direction pairs (θ_k, θ_l), then trajectories emanating between them in the θ_k direction terminate between them in the θ_l direction. Consequently if two Q-domains have the same directions we may find two which have boundary geodesics Γ_1 and Γ_2 which coincide at each end. Consequently $\Gamma_1-\Gamma_2$ would be by (2.10) a closed geodesic, which contradicts (3.6).

We may view the θ_k as the vertices or 0-simplexes of a regular $(m+1)$-polygon in the plane; the directions (θ_k, θ_l) may be viewed as the sides or 1-simplexes. In such a geometric complex there can be at most $2m-1$ sides. Since by (2.7) the $m+1$ sides $(\theta_1, \theta_2), \ldots, (\theta_m, \theta_{m+1}), (\theta_{m+1}, \theta_1)$ correspond to the end domains, there can be at most $m-2$ strip domains S_j, whose directions $(\theta_{k(j)}, \theta_{l(j)})$ satisfy the conditions $l(j) \geqq (k(j)+2) \pmod{(m+1)}$. Examples show that all possible geometric complexes which contain the sides of the regular polygon occur.

(d) The geodesic $g(z_1, z_2)$ which connects z_1 to z_2 is unique. In this section we define a canonical and rigid homotopy $F(t, s) = g_s(t)$ which deforms $g_0(t)$ into $g_\theta(t)$. We shall refer to it by saying we "*bend the right* (*or left*) *end* (*of the geodesic* g) θ *radians toward the left* (*or right*) *side of* g". In fact $F(t, s) = g_s(t)$ will be a geodesic for each fixed s and one end of $g = g_0$ will be rigid ($F(t, s) \equiv F(t, 0)$ for $t \geqq t_0$, $0 \leqq s \leqq \theta$). Of course left and right refers to an orientation of g and radians are measured by the change in arg $[\zeta(F(t_1, s)) - \zeta_0]$ as s varies.

Let g be a geodesic of finite or infinite length. Lemma (2.3) implies that all ζ-angles on the left (or right) side of g are at least π radians. If all ζ-angles on the left (right) side are equal to π then we say that g *cannot be bent toward its left* (*right*) *side.* If some point z_1 exists where the left ζ-angle exceeds π then by (2.5) z_1 must be a zero of $d\zeta^2$. We may assume that z_1 is the farthest point left of such points. We write $g = h_0 + g_1$, where h_0 are the points of g left of z_1 and g_1 are the points of g right of z_1. By (2.6), arg $(\zeta(h_0(t)) - \zeta(z_1))$ is constant on int h_0. In fact ζ can be defined conformally on a neighborhood of int h_0 and $h_0 = \zeta^{-1}(\zeta(z_1) + te^{i\alpha})$ for $t \geqq 0$. We then set

$$h_s(t) = \zeta^{-1}(\zeta(z_1) + te^{i(\alpha - s)})$$

for $t \geqq 0$, s small, and define

$$F(t, s) = g_s(t) = g_s = h_s + g_1.$$

If this is defined for $0 \leqq s \leqq \varepsilon$ then g_ε is the result of *bending the left end of* g, ε *radians toward the left.* As s increases, the left ζ-angle at z_1 decreases. Therefore g_s will be a geodesic by (2.3) as s increases until either

(i) the ζ-angle of g_s on the left at z_1 equals π, or

(ii) int h_s contains a zero of $d\zeta^2$.

In case (i) we search to the right of z_1 for the first zero whose left ζ-angle exceeds π. If none can be found then all left angles are π and g_s cannot be bent farther toward the left. If one can be found, say z_2, then we may repeat the process from the pivot z_2, adding the new radians to the ones already accumulated.

If case (ii) occurs then we let z_2 be the farthest interior zero of int h_s, left on h_s. Since the right ζ-angle at z_2 must be π, the left ζ-angle must be $(e(z_2)+1)\pi$ and we may repeat the process using z_2 as the new pivot and add the new radians to the total radians accumulated already.

(e) Next we consider the half plane domains

$$H_k = \{(-1)^{k-1} \operatorname{Re} \zeta > d_k\} \qquad (1 \leqq k \leqq m+1)$$

which correspond to the horizontal asymptotic direction θ_k of (3.9). Let g be a geodesic with ends of infinite length with one end in H_k and the other in H_l. We may assume that $k<l$ and that g is sensed left to right as we run from H_l to H_k. Consider the left side of g. If all its ζ-angles are π then there are vertical translates of g which are unobstructed geodesics, namely $\zeta^{-1}(\zeta(g(t))+is)$ $(-\infty\leqq t\leqq\infty)$, $0\leqq s<s_1$, and similarly on the right side for $0\geqq s>s_2$. We suppose s_1 and s_2 are the largest and smallest such numbers. Clearly both $s_1=+\infty$ and $s_2=-\infty$ is only possible when $Q(z)\,dz^2\equiv\alpha_{m-1}\,dz^2$, in which case there are no proper end domains. Otherwise there are 3 possible cases:

(i) $\max(|s_1|, |s_2|)=\infty$,

(ii) $0<\max(|s_1|, |s_2|)<\infty$,

(iii) $0=\max(|s_1|, |s_2|)$.

In the first case (i) we say that H_k, H_l are *end connected.* The translates of g span a set K on which ζ is defined conformally and maps onto a half plane $\{\operatorname{Im} e^{i\alpha}\zeta>c\}$. Since $d\zeta^2$ has only $(m+3)-4$ zeros this implies that there is a horizontal end domain $\{\operatorname{Im}\zeta>c\}$ which meets both H_k and H_l. Hence by (2.7), $|k-l|\equiv 1 \bmod (m+1)$.

In the second case (ii) there is a strip $\{|\operatorname{Im} e^{i\alpha}(\zeta-\zeta_0)|<w/2\}$ parallel to one side of g which meets both H_k and H_l. We call this pair H_k, H_l *strip connected.* There are actually two subcases which are important:

(iia) $\{|\operatorname{Im}(\zeta-\zeta_0)|<w/2\}$ meets H_k and H_l,

(iib) not (iia),

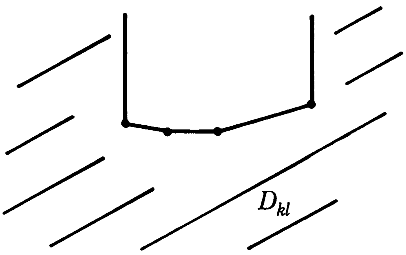
according to whether or not a horizontal strip connects H_k to H_l.

The remaining case (iii) is called *line connected.* No unobstructed geodesic meets both H_k and H_l.

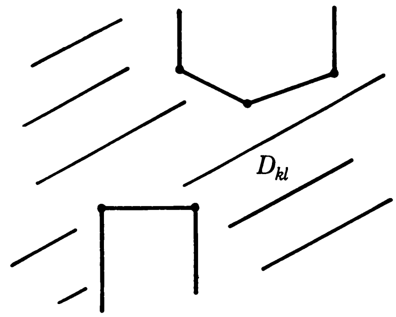
Next we define a closed set D_{kl} for each pair, called the *connecting domain.* In the end connected case (i) we bend the ends of ∂E, the boundary of the connecting end domain E, back toward the vertical, that is, until the ends are vertical geodesics. The *connecting domain* D_{kl} is the union of E and the deformations of ∂E. In the strip connected case we bend the ends of ∂S, the boundary of a connecting strip S, away from S toward the vertical. The *connecting domain* D_{kl} is the union of S and these deformations of ∂S. Finally in the line connected case we may bend the ends of the connecting geodesic g back toward the vertical. The connecting domain D_{kl} is the union of these deformations of g.

In cases (i), (ii) it is possible to define a conformal branch of the Q-integral, say ω, on int D_{kl}, because int D_{kl} is simply connected and critical point free. The image $\omega(D_{kl})$ is diagramed as the shaded area in (3.11) Figures 1, 2a, b. In case (iii), int D_{kl} consists of 2 components both simply connected and joined by the connecting geodesic g. Here ω denotes a branch of the Q-integral defined conformally in one component and extended along one side of g into the other. The case where the images of the components do not overlap is diagramed as the shaded area in (3.11) Figure 3.

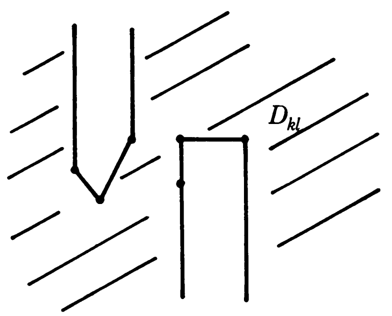
1. End connected



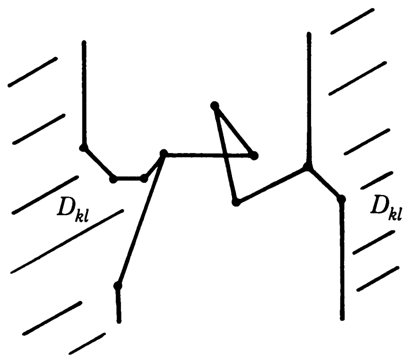
2a. Strip connected



(3.11) 2b. Strip connected



3. Line connected



4. **Horizontal direction adjustment constants.** (a) We begin by studying ζ a fixed branch of the Q-integral. Let V denote a fixed geodesic, say vertical, with precisely one end emanating from ∞. The local trajectory structure of $d\zeta^2$ at ∞ implies that, for $r \geqq r_0$, $U = \{|z| > r\} - V$ is simply connected and critical point free. Hence $\zeta = \int Q^{1/2}\, dz$ may be defined as a locally conformal analytic function on U.

End domains E_k of $d\zeta^2$ and end domains H_k of $-d\zeta^2$ for $1 \leqq k \leqq m+1$ will be called *horizontal*, *vertical half plane domains* respectively. We may assume V lies in E_{m+1}. Then the restriction of the direct analytic continuation of ζ to these domains maps them conformally onto

$$\begin{aligned} \zeta(H_k) &= \{(-1)^{k+1} \operatorname{Re} \zeta > a_k\}, \\ \zeta(E_k) &= \{(-1)^{k+1} \operatorname{Im} \zeta > b_k\}, \end{aligned} \tag{4.1}$$

for $1 \leqq k \leqq m+1$.

Henceforth we reserve ζ to refer to this fixed branch of the Q-integral. We shall use the fact that since $d\zeta^2$ is a quadratic differential, any other branch ω of the Q-integral is related to ζ by an equation of the form

$$\omega = \pm \zeta + d. \tag{4.2}$$

We call d a *quasi-period*.

We wish to expand $\zeta(z)$ in U. Hence we observe from the canonical form (3.1) for $Q(z)$ that

$$Q(z)^{1/2} = \alpha_{m-1}^{1/2} z^{(m-1)/2}\left[1+\sum_{\nu=1}^{m-1}\beta_\nu z^{-\nu}\right]^{1/2}$$
$$= \alpha_{m-1}^{1/2} z^{(m-1)/2}\left[1+\sum_{j=1}^{\infty}\binom{1/2}{j}\left(\sum_{\nu=1}^{m-1}\beta_\nu z^{-\nu}\right)^{j}\right]$$
$$= \alpha_{m-1}^{1/2} z^{(m-1)/2}\left[1+\sum_{j=1}^{\infty}\binom{1/2}{j}\sum_{S(j)}\binom{j}{k_1,\ldots,k_{m-1}}\prod_{\nu=1}^{m-1}(\beta_\nu z^{-\nu})^{k_\nu}\right]$$

where

$$S(j) = \{k_1,\ldots,k_{m-1} : k_\nu \geqq 0, k_1+\cdots+k_{m-1} = j\},$$
$$\binom{1/2}{j} = \frac{(1/2)(1/2-1)\cdots(1/2-j+1)}{j!},$$
$$\binom{j}{k_1,\ldots,k_{m-1}} = \frac{j!}{k_1!k_2!\cdots k_{m-1}!}.$$

Hence after collecting the coefficients belonging to the powers (possibly fractional) of z and integrating term by term we obtain the following desired expansion:

$$\zeta = {\sum_{n\geqq 0}}' \sigma_n z^{(m+1)/2-n} + \sigma_l \log z + \sigma_c, \tag{4.3}$$

where prime denotes the absence of a constant term (when $m+1$ is even), $\sigma_l=0$ when $m+1$ is odd, and σ_c is the constant of integration. In addition we have a formula for the coefficients

$$\sigma_n = \frac{2\alpha^{1/2}}{(m+1-2n)}\left[\sum_{I(n)}\binom{1/2}{k_1,\ldots,k_{m-1}}\prod_{\nu=1}^{m-1}\beta_\nu^{k_\nu}\right], \tag{4.4}$$

where

$$\binom{1/2}{0,0,\ldots,0} = 1,$$
$$I(n) = \{(k_1,\ldots,k_{m-1}) : k_\nu \geqq 0, 1k_1+2k_2+\cdots+(m-1)k_{m-1} = n\},$$
$$\binom{1/2}{k_1,\ldots,k_{m-1}} = \frac{(1/2)(1/2-1)\cdots(1/2-k_1-\cdots-k_{m-1}+1)}{k_1!k_2!\cdots k_{m-1}!}.$$

Notice that $k_\nu=0$ for $\nu>n$ in $I(n)$. Also, this formula holds for σ_l when $(m+1)$ is even, provided that we employ the conventions $l=n=(m+1)/2$ and $2/(m+1-2l)=1$ in that case.

From expansion (4.3) we note that if z, $e^{2\pi i}z$ represent the prime ends of $U-V$ at $z\in V$, then

$$\zeta(e^{2\pi i}z) = e^{(m+1)\pi i}(\zeta(z)+q), \tag{4.5}$$

where the quasi-period q satisfies

$$q = 2\pi i\sigma_l + (e^{(m+1)\pi i}-1)\sigma_c. \tag{4.6}$$

In other words $q=2\pi i\sigma_l$ when $m+1$ is even and $q=-2\sigma_c$ when $m+1$ is odd.

(b) Next we define some canonical parametrized exhaustions. Let γ denote any Jordan curve on the sphere, not containing the point ∞. Then U_γ, J_γ denote the Jordan domains bounded by γ containing, not containing ∞, respectively. They are also of infinite, finite $|d\zeta|^2$-density respectively.

Recall now that ζ is defined on $U=\{|z|>r\}-V$, for a vertical geodesic V. Then let $\gamma_1=\gamma_1(r;\zeta)=\{z\in U : |\zeta(z)|=r\}$ and let $\gamma_2=\gamma_2(r;\zeta)$ equal the geodesic on V which joins the end points of γ_1. Finally we define $\gamma_r=\gamma_1+\gamma_2$. Evidently we have

$$\zeta(\gamma_1(\theta)) = re^{i\theta} \tag{4.7}$$

for $\theta_1(r)\leqq\theta\leqq\theta_2(r)$, where

$$\begin{aligned} \theta_1(r) &= -\frac{\pi}{2}+\sin^{-1}\frac{a_1}{r}, \\ \theta_2(r) &= -\frac{\pi}{2}+(m+1)\pi+e^{(m+1)\pi i}\sin^{-1}\left(\frac{a_1+\operatorname{Re} q}{r}\right), \end{aligned} \tag{4.8}$$

provided V is on the right boundary of H_1 (see (4.1)).

(c) The ends of all horizontal geodesics of infinite length tend to the point ∞ in one of the $m+1$ horizontal directions θ_k of (3.9) for $1\leqq k\leqq m+1$. The direction θ_k corresponds to the vertical half plane H_k. Now $\gamma_1(r;\zeta)\cap H_k$ contains a unique point $x_k=x_k(r,\zeta)$ for which $\zeta(x_k)=(-1)^{k-1}r$. We shall prove that

$$\int_{g(x_k,x_j)}|d\zeta| = 2r+h_{jk}+o(1). \tag{4.9}$$

The constants $h_{jk}=h_{jk}(\zeta)$ clearly depend on the branch ζ and will be called the *horizontal direction adjustment constants* (hdac) *of* ζ. However, when ζ is fixed we shall suppress its dependence on ζ in our notation.

We require a slightly stronger statement than (4.9). Let z_k denote an arbitrary point in H_k. Set

$$z_k(t) = \zeta^{-1}(\zeta(z_k)+(-1)^{k-1}t) \tag{4.10}$$

for $t\geqq 0$, $1\leqq k\leqq m+1$. Then if $g(z,w)$ denotes the geodesic with end points z and w and $g(t)=g(z_k(t),z_j(t))$, we prove

LEMMA (4.11). *Under the preceding notation*

$$\int_{g(z_k(t),z_{k+1}(t))}|d\zeta| \geqq (-1)^{k-1}\operatorname{Re}\zeta(z_k(t))+(-1)^k\operatorname{Re}\zeta(z_{k+1}(t)),$$

$$\int_{g(z_1(t),z_{m+1}(t))}|d\zeta| \geqq \operatorname{Re}\zeta(z_1(t))+(-1)^m\operatorname{Re}\zeta(z_{m+1}(t))+\operatorname{Re} q,$$

in directions of end domains, whereas in the other directions

$$\int_{g(z_k(t),z_j(t))}|d\zeta| = (-1)^{k-1}\operatorname{Re}\zeta(z_k(t))+(-1)^{j-1}\operatorname{Re}\zeta(z_j(t))+h_{jk}+o(1).$$

Proof. The idea is to use the connection domain D and Lemma (2.3) to construct the geodesic $g(t)$ explicitly. The Lemma (4.11) is then easily verified in each case.

Suppose first that H_j, H_k are end connected and $k=(j+1)$ for $1 \leqq j \leqq m$. For definiteness we assume that $\zeta(E_j)$ is an upper half plane and $\zeta(E_j)$ is sensed left to right as we move from H_{j+1} into H_j. Let z_1, z_2 be the farthest zeros left, right on ∂E_j, respectively. Then set

$$G_1 = \{\pi \leqq \arg(\zeta - \zeta(z_1)) \leqq \pi + \varepsilon\},$$
$$G_2 = \{0 \geqq \arg(\zeta - \zeta(z_2)) \geqq \varepsilon\},$$

where $\pi+\varepsilon$ is smaller than the minimum of the two ζ-angles on the side of E_j at z_1, z_2. Then set

$$D_\varepsilon = \operatorname{cl} E_j \cup G_1 \cup G_2.$$

For $t \geqq t_0$, $z_j(t)$, $z_{j+1}(t) \in D_\varepsilon$. It is now clear from (3.11) Figure 1 and Lemma (2.3) that the geodesic $g(t)$ consists of 1, 2 or 3 geodesics whose interiors are critical point free. If ω is the conformal map on $D_{j,j+1}$ then

$$\int_{g(t)} |d\zeta| \geqq (-1)^{j-1} \operatorname{Re} \omega(z_j(t)) + (-1)^j \operatorname{Re} \omega(z_{j+1}(t)).$$

If $1 \leqq j \leqq m$ then $\omega = \zeta$ and

$$h_{j,j+1} = 0. \tag{4.12}$$

If $j=m$ then $\omega=\zeta$ in H_{m+1} continues analytically through E_{m+1} to the branch $\omega(z)=\zeta(e^{2\pi i}z)$ in H_1, and hence by (4.5) we have

$$h_{1,m+1} = \operatorname{Re} q. \tag{4.13}$$

Next suppose that H_j, H_k are strip connected. Suppose first that there is a horizontal connecting strip, critical point free, of maximum width

$$S = \{|\operatorname{Im}(\zeta - \zeta_0)| < w/2\}$$

which connects H_j to H_k. That is, S is a strip domain in the global trajectory structure of $d\zeta^2$. Let us sense S left to right from H_j into H_k and let ω be the analytic continuation of ζ in H_k into H_j through S. It makes sense to speak of the upper and lower boundaries L_1, L_2 of S. Let z_1, z_2 be the first and last zeros on L_1, let z_3, z_4 be the first and last zeros L_2. Define 4 angular domains $G_1, \ldots, G_4$ at these points similar to the domains G_1, G_2 in the last case and set $D_\varepsilon = \operatorname{cl} S \cup \bigcup_1^4 G_v$. Then again for $t \geqq t_0$, $z_j(t)$, $z_k(t) \in D_\varepsilon$, and from (3.11) Figure 2a and Lemma (2.3), it is clear that $g(t)$ consists of at most 3 geodesics. The outside geodesics have critical point free interiors. The middle horizontal geodesic is independent of t. Consequently we have

$$\int_{g(t)} |d\zeta| = (-1)^{k-1}[\operatorname{Re} \zeta(z_k(t)) - \operatorname{Re} \omega(z_j(t))] + o(1).$$

But (4.2) implies that $\omega = \pm\zeta + d$ in H_j and the fact that $(-1)^{k-1}\operatorname{Re}\omega(z_j(t)) < 0$ will tell us the sign. Hence (4.11) holds in this case and

$$h_{jk} = -\operatorname{Re} d. \tag{4.14}$$

If on the other hand H_j, H_k are strip connected by

$$S = \{|\operatorname{Im} e^{-i\theta}(\zeta-\zeta_0)| < w/2\}$$

but not by a horizontal strip, then we know that $0 < |\theta| < \pi/2$ and for definiteness we assume $\theta > 0$. We may also assume that S is sensed as before and L_1, L_2, ω are defined as before. We want to define new points instead of z_1, z_2, z_3, z_4. The deformations of L_1, L_2 form a lower and upper boundary of the connecting domain D_{jk}, say L and U (see (3.11) Figure 2b). Let z_4 be the farthest point right on L for which $\sup_L \operatorname{Im}\zeta(z)$ is attained and z_1 the farthest point left on U for which $\inf_U \operatorname{Im}\zeta(z)$ is attained. Evidently, no horizontal connecting strips implies

$$\operatorname{Im}\zeta(z_1) \leqq \operatorname{Im}\zeta(z_4).$$

If $\operatorname{Im}\zeta(z_1) = \operatorname{Im}\zeta(z_4)$ then we define z_2 as the farthest point right on U for which $\operatorname{Im}\zeta(z_1) = \operatorname{Im}\zeta(z_2)$ and z_3 as the farthest point left on L for which $\operatorname{Im}\zeta(z_3) = \operatorname{Im}\zeta(z_4)$. The previous argument for horizontal connecting strips is then repeated.

If $\operatorname{Im}\zeta(z_1) < \operatorname{Im}\zeta(z_4)$ then z_1 and z_4 are called the *contact points*. Evidently the angular domains

$$G_1 = \{|\arg(\omega-\omega(z_4))| \leqq \varepsilon\},$$
$$G_2 = \{|\arg(\omega-\omega(z_1))+\pi| \leqq \varepsilon\}$$

are critical point free for ε small enough and the domain

$$D_\varepsilon = G_1 \cup g(z_1, z_4) \cup G_4$$

contains $z_j(t)$, $z_k(t)$ for $t \geqq t_0$. Hence by Lemma (2.3) we have

$$g(z_j(t), z_k(t)) = g(z_j(t), z_1) + g(z_1, z_4) + g(z_4, z_k(t)).$$

We approximate the lengths of the first and last geodesics as before to obtain

$$\int_{g(t)} |d\zeta|$$
$$= (-1)^{k-1}[\operatorname{Re}\zeta(z_k(t)) - \operatorname{Re}\zeta(z_4) - \operatorname{Re}\omega(z_j(t)) + \operatorname{Re}\omega(z_1)] + \int_{g(z_1,z_4)} |d\zeta| + o(1).$$

Again $\omega = \pm\zeta + d$ in H_j and $(-1)^{k-1}\operatorname{Re}\omega(z_j(t)) < 0$ implies that $\omega = (-1)^{j-1}\zeta + d$ in H_j. Hence we find that (4.11) holds and

$$h_{jk} = (-1)^k \operatorname{Re}\zeta(z_4) + (-1)^j \operatorname{Re}\zeta(z_1) + \int_{g(z_1,z_4)} |d\zeta|. \tag{4.15}$$

Finally let us consider the last case that H_j is line connected to H_k. Let g be a connecting geodesic of infinite length in both directions sensed left to right from H_j to H_k. Let U, L denote the deformations obtained by bending the ends of g toward the vertical. In this case $U \cap L = g(z_1, z_2)$, where z_1 is left of z_2. Clearly ζ may be defined conformally on both components of the interior of the connecting domain D_{kl} by direct analytic continuations. Let $w_1, \ldots, w_\mu = z_1$ be the zeros on U left of z_1 and $v_1, \ldots, v_\nu = z_1$ the zeros on L left of z_1. For definiteness assume Im ζ increases, decreases as we move left on the left vertical end of U, L respectively.

As we move left on the initial segments $g(w_{\mu-1}, w_\mu)$, $g(v_{\nu-1}, v_\nu)$ either we move up on the first or we move down on the second since they have disjoint interiors. For definiteness we assume we move down on $g(v_{\nu-1}, v_\nu)$. Then let z_{jk} be the farthest point right on U, but not right of z_1 on which $\inf \operatorname{Im} \zeta(z)$ is obtained for points $z \in U$ left of z_1. We call z_{jk} the *contact* point of H_j to H_k. Let w_{jk} be the farthest point left on U for which this infimum is obtained. Evidently $w_{jk} = z_{jk}$ unless there are some horizontal geodesics left of z_{jk} on U. Similarly there must be a contact point z_{kj} at H_k and a corresponding w_{kj}. Now we form 4 angular domains G_1, G_2, G_3, G_4 from $z_{jk}, w_{jk}, z_{kj}, w_{kj}$ as before and set

$$D_\varepsilon = g(z_{jk}, z_{kj}) \cup \bigcup_1^4 G_\nu.$$

Clearly for $t \geqq t_0$, $z_j(t), z_k(t) \in D_\varepsilon$ and $g(z_j(t), z_k(t))$ can be written as 5 geodesics

$$g(t) = g(z_j(t), w_{jk}) + g(w_{jk}, z_{jk}) + g(z_{jk}, z_{kj}) + g(z_{kj}, w_{kj}) + g(w_{kj}, z_k(t)).$$

The first and last can be estimated asymptotically as $t \to \infty$ and we observe as before that (4.11) holds with the formula

$$h_{jk} = (-1)^j \operatorname{Re} \zeta(z_{jk}) + (-1)^k \operatorname{Re} \zeta(z_{kj}) + \int_{g(z_{jk}, z_{kj})} |d\zeta|. \tag{4.16}$$

Hence (4.11) holds in all cases. As a corollary we observe that the proof gives us a way to compute the hdac h_{jk}. In addition the following interesting relation is clear from the proof.

COROLLARY (4.17). *If* $\omega = \zeta + d$ *then* $h_{jk}(\omega) = h_{jk}(\zeta) + ((-1)^{k+1} + (-1)^{j+1}) \operatorname{Re} d$.

(d) In §4(b) we defined ζ on $U = \{|z| > r\} - V$ and then $\gamma_r = \gamma_1 + \gamma_2$ where $\gamma_2 \subset V$. Now V was chosen as a vertical geodesic so that the horizontal direction adjustment constant h_{1k} would be well defined. However, for our proof it will be necessary or at least easier if we take V to be a horizontal geodesic. The a_k, b_k of (4.1) are the same as before. Define J_r to be the domain bounded by γ_r which does not contain ∞. Since we want to estimate $\iint_{J_r} |d\zeta|^2$ asymptotically the following relation is useful.

LEMMA (4.18).

$$2\left(\sum_1^{m-2} w_j - \sum_1^{m+1} b_k\right) = \operatorname{Im} q.$$

We recall that b_k is defined in (4.1), w_j is the width of the jth strip domain and q is the quasi-period (4.6). To prove it we observe from (4.5), (4.7) that

$$\int_{\gamma_r} |d\operatorname{Im}\zeta| = 2(m+1)r+\operatorname{Im} q+o(1). \tag{4.19}$$

In strip domains S_j and end domains E_k,

$$\int_{\gamma_r\cap E_k} |d\operatorname{Im}\zeta| = 2(r-b_k), \qquad \int_{S_j\cap\gamma_r} |d\operatorname{Im}\zeta| = 2w_j.$$

Hence if we add these we obtain

$$\int_{\gamma_r} |d\operatorname{Im}\zeta| = 2(m+1)r+2\left(\sum_1^{m-2} w_j-\sum_1^{m+1} b_k\right) \tag{4.20}$$

and consequently comparison of (4.19) with (4.20) yields (4.18).

We underline the fact that the choice of ζ is fixed throughout this discussion. However, it is interesting to note that although the values b_k, q depend on the choice of ζ, the values $2\sum_1^{m+1} b_k+\operatorname{Im} q$ does not.

LEMMA (4.21).

$$\iint_{J_r} |d\zeta|^2 = \frac{(m+1)\pi}{2} r^2+r\operatorname{Im} q+\sum_1^{m-2} h_j w_j+o(1),$$

where h_j is the hdac of the directions defined by the jth strip domain and w_j is its width.

Proof. Let $\{\Delta\}$ denote the end and strip domain in the global trajectory structure of $d\zeta^2$. Define $\Delta_r=\Delta\cap J_r$ and notice that

$$\iint_{J_r} |d\zeta|^2 = \sum_{\Delta} \iint_{\Delta_r} |d\zeta|^2.$$

Clearly we have

$$\iint_{E_k} |d\zeta|^2 = \frac{\pi}{2} r^2-2rb_k+O(1),$$

$$\iint_{S_j} |d\zeta|^2 = (2r+h_j)w_j+o(1).$$

Adding these we have

$$\iint_{J_r} |d\zeta|^2 = (m+1)\frac{\pi}{2} r^2+2\left(\sum_1^{m-2} w_j-\sum_1^{m+1} b_k\right)r+\sum_1^{m-2} h_j w_j+o(1).$$

Now (4.18) implies (4.21).

5. **The extended inequality.** (a) The inequalities of Teichmüller and Jenkins represent the limit of the difference between the estimates (above and below) of the change in area of $D_r = D \cap J_r$ (measured by the density $|d\zeta|^2$) under the mapping by h. The estimates are made by the length-area technique. In this section we shall describe a procedure for overcoming the difficulty in applying the length-area technique when h does not have a coefficient normalization.

Let $(d\zeta^2, R; h, D)$ be a given situation in Teichmüller's case and let ζ be a fixed branch of the Q-integral on $U = \{|z| > r\} - V$, for a horizontal geodesic V on the right end of the boundary of E_1. Let ω be any branch of the Q-integral defined on $h(U)$. The canonical curves $\gamma_r = \gamma(r; \zeta)$ and exhaustions J_r, D_r, Δ_r are then defined. We have by (4.3), (3.2) the following expansions:

$$\zeta(z) = \sum_{n \geqq 0}' \sigma_n z^{(m+1)/2-n} + \sigma_l \log z + \sigma_c, \tag{5.1}$$

$$\omega(w) = \sum_{n \geqq 0} \lambda_n w^{(m+1)/2-n} + \lambda_l \log w + \lambda_c, \tag{5.2}$$

$$w = h(z) = z + \sum_{n=0}^{\infty} c_n z^{-n} \qquad (c_0 = 0). \tag{5.3}$$

In this case $\lambda_n = \sigma_n$, $\lambda_l = \sigma_l$. Then we have the expansion for the composition

$$\omega(h(z)) = \sum_{n \geqq 0}' \tau_n z^{(m+1)/2-n} + \dot{\tau}_l \log z + \tau_c \tag{5.4}$$

where $\lambda_l = \tau_l$ and

$$\tau_n = \sum_{\nu=0}^{n} \lambda_\nu c_{n-\nu}^{((m+1)/2-\nu)}, \tag{5.5}$$

with the following conventions. First when m is odd $\lambda_{(m+1)/2} = \lambda_l$. Second

$$\begin{aligned} [h(z)]^\alpha &= \sum_{k=0}^{\infty} c_k^{(\alpha)} z^{\alpha-k} && (\alpha \neq 0), \\ \lambda_c + \log h(z) &= \log z + \sum_{k=0}^{\infty} c_k^{(0)} z^{\alpha-k} && (\alpha = 0). \end{aligned} \tag{5.6}$$

Note that $c_0^{(\alpha)} = 1$ if $\alpha \neq 0$. For $n \geqq 1$ we have the following formula for $c_n^{(\alpha)}$:

$$c_n^{(\alpha)} = \sum_{1k_1 + \cdots + nk_n = n; k_\nu \geqq 0} \left[\binom{\alpha}{k_1 + k_2 + \cdots + k_n} \prod_{\nu=1}^{n} (c_{\nu-1})^{k_\nu} \right] \tag{5.7}$$

where

$$\begin{aligned} \binom{\alpha}{n} &= \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} && \text{if } \alpha \text{ is not a nonnegative integer,} \\ &= \frac{(-1)^{n-1}}{n} && \text{if } \alpha = 0, \\ &= \frac{\alpha!}{n!(\alpha-n)!} && \text{if } \alpha \text{ pos. int., } n \leqq \alpha, \\ &= 0 && \text{if } \alpha \text{ pos. int., } n > \alpha, \end{aligned}$$

for $n \geqq 1$. It is clear from these formulas that if $\lambda_n = \sigma_n$ and Jenkins' normalization (3.5) holds then $\tau_n = \sigma_n$ for $n \leqq [m/2]$ and $\sigma_l = \tau_l$. Therefore $H = \zeta \circ h \circ \zeta^{-1}$ on $\zeta(U)$ will have the following normalized expansion:

$$H(\zeta) = \zeta + \sum_{0 \geqq t \geqq -1} \mu_t \zeta^t + o(|\zeta|^{-1}) \tag{5.8}$$

where either $t=0$ or $t=1-2n/(m+1)$ for $(m+1)/2 \leqq n \leqq m+1$. We remark that the actual order of the remainder term is $O(|\zeta|^{-(m+3)/(m+1)})$ when m is even and $O(|\zeta|^{-(m+3)/(m+1)} \log |\zeta|)$ when m is odd. Both of these are $o(|\zeta|^{-1})$.

The key observation is that Jenkins' proof depends on the normalization (5.8) of H rather than the normalization (3.5) of h. Let us expand on this. Jenkins' result is obtained by estimating the difference in the Q-densities of $h(D_r)$ and D_r. That is,

$$l(r) \leqq \iint_{h(D_r)} |d\zeta|^2 - \iint_{D_r} |d\zeta|^2 \leqq u(r).$$

The estimates $u(r)$, $l(r)$ are in terms of the coefficients of H. They are obtained by using essentially Schwarz's inequality to compare length to area. If they are so precise that

$$u(r) - l(r) = A + o(1), \tag{5.9}$$

then the result $A \geqq 0$ can be concluded. Expansion (5.8) of H and Jenkins' technique of estimation gives (5.9). However, if fewer initial coefficients of h are nonzero than in Jenkins' normalization (3.5) then H will have some nonzero coefficients on its positive fractional terms and

$$u(r) - l(r) = K|\mu_{t_0}|^2 r^{2t} + o(r^{2t}). \tag{5.10}$$

The result $A \geqq 0$ is not obtained since the difference diverges.

(b) To overcome this obstacle we change the point of view. First the estimates are essentially on $\iint_{h(D_r)} |d\zeta|^2$ rather than the difference. (Estimation of the difference merely simplifies computation.) Hence we estimate the area of $h(D_r)$. Second, we choose a new density $|d\omega|^2$ of the same type as $|d\zeta|^2$ and estimate the $|d\omega|^2$-density of $h(D_r)$. The choice of $d\omega^2$ is made so that when a branch of its integral ω is used to form $H = \omega \circ h \circ \zeta^{-1}$, then H will have the expansion (5.8) and the length-area estimates will give (5.9).

(c) The actual normalization we need will require the *additional restriction* $\operatorname{Im} \mu_0 = 0$ when $m+1$ is odd. We shall call $(d\omega^2, \omega)$ an *allowable pair* if $d\omega^2$ is of the form (3.1) and $H = \omega \circ h \circ \zeta^{-1}$ has expansion (5.8) with the additional restriction. We now prove the existence of allowable pairs.

LEMMA (5.11). *For each situation $(d\zeta^2, R; h, D)$ and choice of ζ in Teichmüller's case there is a space of allowable pairs $(d\omega^2, \omega)$ which can be parametrized by $m-1$ real variables.*

Proof. We consider $d\omega^2 = P(w)\, dw^2$ of form (3.1). Then $\omega(h(z))$ has expansion (5.4) where the coefficients are given by formulas (5.6), (5.7) and (4.4). We compare this to the expansion (5.1) for $\zeta(z)$ Then $\omega \circ h \circ \zeta^{-1} = H$ will have the desired expansion if and only if

$$(5.12)\qquad \begin{aligned} \sigma_n &= \tau_n && (\text{for } 0 \leqq n \leqq [m/2]),\\ \sigma_l &= \tau_l && (\text{for } m \text{ odd}),\\ \operatorname{Im} \sigma_c &= \operatorname{Im} \tau_c && (\text{for } m \text{ even}). \end{aligned}$$

Now let $d\omega^2 = P(w)\, dw^2 = \gamma w^{m-1}(1+\sum_{\nu=1}^{m-1} \delta_\nu w^{-\nu})\, dw^2$. Then $\sigma_0 = \tau_0$ if and only if $\alpha = \gamma$. By formulas (4.4) and (5.5) it is clear that

$$\tau_n = K\alpha^{1/2}\delta_n + G(\alpha, \delta_1, \ldots, \delta_{n-1}; h).$$

Since we have $\alpha \neq 0$ it is clear that (5.12) holds for a unique choice of γ and $\delta_1, \ldots, \delta_{[(m+1)/2]}$ and, in the even case, the requirement $\operatorname{Im} \sigma_c = \operatorname{Im} \tau_c$. Hence $m+3$ real variables are required. Counting the constant of integration there are $2(m+1)$ real variables available, hence $2(m+1)-(m+3)=(m-1)$ remain free to parametrize the space $\{(d\omega^2, \omega)\}$.

COROLLARY (5.13). *If $(d\omega^2, \omega)$ is allowable then*

(i) *$d\omega^2$ has the same horizontal directions θ_k as $d\zeta^2$.*

(ii) *ω has the same quasi-period as ζ if $(m+1)$ is even and the same imaginary part of its quasi-period as ζ if $m+1$ is odd.*

This follows from (5.12).

(d) We now state our main result.

THEOREM (5.14) (COMPLETE TEICHMÜLLER INEQUALITIES). *For each situation $(d\zeta^2, R; h, D)$ in Teichmüller's case and choice of ζ let $(d\omega^2, \omega)$ be allowable. Then*

$$(5.15)\qquad \begin{aligned} &(m+1)\pi \operatorname{Re} \mu_{-1} + \operatorname{Im} (\mu_0 q) + 2\sum_1^{m-2} A_j(\omega)c_j(\zeta) \\ &\qquad \leqq \sum_1^{m-2} A_j(\zeta)c_j(\zeta) + \sum_1^{m-2} B_j(\omega)\, d_j(\omega), \end{aligned}$$

where the constants in (5.15) *are listed in the Table of Constants* (5.16).

Equality occurs if and only if h is a horizontal isometry, in other words if and only if $h = \omega^{-1} \circ \zeta$ on an open subset of D.

(5.16) *Table of Constants.*

μ_0, μ_{-1} are the constants in the expansion (5.8) of $H = \omega \circ f \circ \zeta^{-1}$.

q is the quasi-period of ω.

$c_j(\zeta), d_j(\omega)$ are the widths of the strip domain in the global trajectory structure of $d\zeta^2, d\omega^2$ respectively.

$A_j(\zeta), B_j(\omega)$ are the hdac (4.9) of ζ, ω corresponding to the directions of the jth strip domain of $d\zeta^2, d\omega^2$ respectively.

$A_j(\omega)$ are the ω-hdac (4.9) of the directions corresponding to the jth strip domain of $d\zeta^2$.

In Teichmüller's and Jenkins' case $(d\zeta^2, \zeta)$ itself is allowable and hence $A_j(\omega) = B_j(\zeta) = A_j(\zeta)$, $c_j(\zeta) = d_j(\omega)$. In Teichmüller's case $\mu_0 = 0$ too. Hence Teichmüller's Theorem is

$$(m+1)\pi \operatorname{Re} \mu_{-1} \leqq 0$$

and Jenkins' Theorem is

$$(m+1)\pi \operatorname{Re} \mu_{-1} + \operatorname{Im}(\mu_0 q) \leqq 0.$$

The proof of Theorem (5.14) is carried out in the remainder of this paper.

6. **The upper estimate** $u(r)$. We have ζ defined on $U=\{|z|>r\}-V$, where V is a horizontal geodesic emanating from ∞ on the right side of the boundary of E_1. The canonical curves $\gamma=\gamma_r=\gamma_r(\zeta)$ and exhaustions J_r, D_r, Δ_r are then defined. We wish to estimate $\iint_{h(D_r)} |d\omega|^2$ from above.

The complement of $h(D_r)$ is a compact set K of finite $|d\omega|^2$-density. Since $J_{h(\gamma)}$ is a disjoint union of $h(D_r)$ and K we may write

$$\iint_{h(D_r)} |d\omega|^2 \leqq \iint_{J_{h(\gamma)}} |d\omega|^2. \tag{6.1}$$

Equality occurs if and only if the $|d\omega|^2$-density of K is zero. That is equivalent to K having Lebesgue density zero. Our estimate $u(r)$ will be an asymptotic expansion of $\iint_{J_{h(\gamma)}} |d\omega|^2$ in terms of our constants. To that end we set $B=\gamma_r(\omega)$ and write

$$\iint_{J_{h(\gamma)}} |d\omega|^2 = \iint_{J_B} |d\omega|^2 + \left(\iint_{J_{h(\gamma)}} |d\omega|^2 - \iint_{J_B} |d\omega|^2\right). \tag{6.2}$$

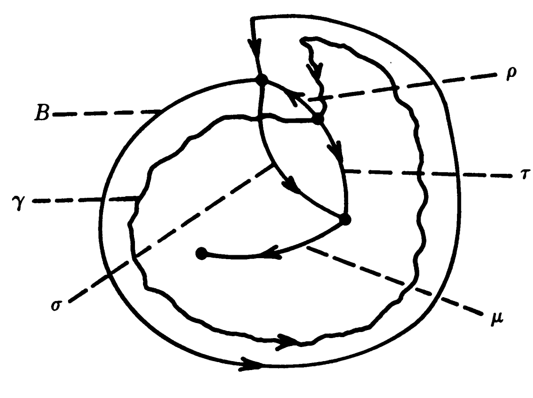
Lemma (4.21) tells us that

$$\iint_{J_B} |d\omega|^2 = \frac{m+1}{2}\pi r^2 + r \operatorname{Im} q + \sum_{j=1}^{m-2} B_j(\omega)\, d_j(\omega) + o(1), \tag{6.3}$$

where $\operatorname{Im} q$ is the common imaginary part of the quasi-periods of ζ and ω, and $B_j(\omega)$, $d_j(\omega)$ are defined in the Table of Constants (5.16).

The task remains to estimate the difference in the $|d\omega|^2$-densities of $J_{h(\gamma)}$ and J_B. This will be accomplished by the use of Stokes' Theorem. Since ω is not defined on these regions we must consider subregions of equal density, where ω can be defined. Technically this is accomplished in the paragraph below. Diagram (6.4) may be used to help keep account of the arcs in the ensuing discussion.

(6.4)



Let $\mu=\mu(t)$ $(0\leqq t\leqq 1)$ denote a smooth Jordan arc in $R-\{\infty\}$ which contains all the zeros of $d\omega^2$. For $r\geqq r_0$, μ will be contained in a Euclidean disk whose closure lies in $J_B\cap J_{h(\gamma)}$. We may assume that r_0 is so large that $h(\gamma(0))$, $B(0)$ lie in a disk whose closure is disjoint from the first disk. Let $\rho=\rho(t)$ $(0\leqq t\leqq 1)$ be a smooth Jordan arc which runs from $h(\gamma(0))$ to $B(0)$ in this disk. Let $\sigma=\sigma(t)$ $(0\leqq t\leqq 1)$ be a smooth Jordan arc in $J_B-\mu$ which runs from $B(0)$ to $\mu(0)$. Finally let $\tau=\tau(t)$ $(0\leqq t\leqq 1)$ be a smooth Jordan arc in $J_{h(\gamma)}-\mu$ which runs from $h(\gamma(0))$ to $\mu(0)$ and which is homotopic to $\rho\sigma$ in $R-\{\infty\}-\mu$.

Since the supports of these arcs μ, σ, τ, ρ have zero $|d\omega|^2$-density we may write

$$\iint_{J_{h(\gamma)}} |d\omega|^2-\iint_{J_B}|d\omega|^2 = \iint_{J_{h(\gamma)}-\tau\mu} |d\omega|^2-\iint_{J_B-\sigma\mu} |d\omega|^2. \tag{6.5}$$

But $J_{h(\gamma)}-\tau\mu$ and $J_B-\sigma\mu$ are simply connected, critical point free domains. Consequently the branch of ω on the intersection of $h(U)$ and the disk containing $B(0)$, $h(\gamma(0))$ may be continued analytically into $J_{h(\gamma)-\tau\mu}$, $J_{B-\sigma\mu}$ to obtain analytic functions ω_2, ω_1 respectively on these regions. They have continuous extensions to the sides ρ_j, σ_j, τ_j, μ_j (for $j=l, r$). Therefore by Stokes' Theorem we have

$$\iint_{J_{h(\gamma)}-\tau\mu} |d\omega|^2-\iint_{J_B-\sigma\mu} |d\omega|^2 = \operatorname{Re}\left(\frac{1}{2i}\int_{C_2}\bar{\omega}_2\, d\omega_2-\frac{1}{2i}\int_{C_1}\bar{\omega}_1\, d\omega_1\right), \tag{6.6}$$

where

$$C_1 = B_1+B_2+\sigma_l+\mu_l-\mu_r-\sigma_r,$$
$$C_2 = h(\gamma_1)+h(\gamma_2)+\tau_l+\mu_l-\mu_r-\tau_r.$$

Consequently, in view of the homotopy between τ and $\rho\sigma$, the quasi-periodicity (4.5) of ω, and cancellation of the integrals over μ_r, μ_l, we may write

$$\begin{aligned}&\operatorname{Re}\frac{1}{2i}\left(\int_{C_2}\bar{\omega}_2\, d\omega_2-\int_{C_1}\bar{\omega}_1\, d\omega_1\right)\\ &= \operatorname{Re}\frac{1}{2i}\int_{\rho_r-\rho_l}\bar{\omega}\, d\omega+\operatorname{Re}\frac{1}{2i}\int_{h(\gamma_1)-B_1}\bar{\omega}\, d\omega+\operatorname{Re}\frac{1}{2i}\int_{h(\gamma_2)-B_2}\bar{\omega}\, d\omega.\end{aligned} \tag{6.7}$$

We must keep in mind which branch of ω is defined on these various arcs. We do this by saying that ρ_l connects $h(\gamma_1(0))$ to $B_1(0)$ and ρ_r connects $h(\gamma_2(1))$ to $B_2(1)$. To estimate the first of these 3 integrals we use (4.5), the quasi-periodicity of ω to write

$$\begin{aligned}\operatorname{Re}\frac{1}{2i}\int_{\rho_r-\rho_l}\bar{\omega}\, d\omega &= \operatorname{Re}\frac{1}{2i}\int_{\rho_l}(\overline{\varepsilon(\omega^+q)}\varepsilon\, d\omega-\bar{\omega}\, d\omega)\\ &= \operatorname{Re}\frac{1}{2i}\bar{q}\int_{\rho_l}d\omega = \operatorname{Re}\left[\frac{1}{2i}\bar{q}(\omega(B(0))-\omega(h(\gamma(0))))\right],\end{aligned} \tag{6.8}$$

where $\varepsilon=(-1)^{m+1}$. But $\omega(B(0))=r$, $\omega(h(\gamma(0)))=H(r)=r+\mu_0+o(1)$ and consequently $-\operatorname{Re}(1/2i)\bar{q}\mu_0=(1/2)\operatorname{Im} q\bar{\mu}_0$ implies

$$\operatorname{Re}\frac{1}{2i}\int_{\rho_r-\rho_l}\bar{\omega}\, d\omega = \frac{1}{2}\operatorname{Im}(q\bar{\mu}_0)+o(1). \tag{6.9}$$

To estimate the second of these 3 integrals in (6.7) recall that $\omega(B_1(\theta))=re^{i\theta}$ for $\theta_1 \leqq \theta \leqq \theta_2$. Therefore

$$\text{(6.10)} \qquad \operatorname{Re} \frac{1}{2i} \int_{h(\gamma_1)-B_1} \bar{\omega}\, d\omega = \operatorname{Re} \frac{1}{2i} \int_{\theta_1}^{\theta_2} (\overline{H(re^{i\theta})}\, dH(re^{i\theta}) - \overline{re^{i\theta}}\, dre^{i\theta}).$$

Now we expand $H(re^{i\theta})$ by (5.8).

$$H(re^{i\theta}) = re^{i\theta} + \mu_0 + \sum \mu_t r^t e^{it\theta} + o(r^{-1}),$$

where sums are indexed by $0 > t \geqq -1$ here. Therefore (6.10) becomes

$$\text{(6.11)} \qquad \begin{aligned} &\operatorname{Re} \frac{1}{2i} \int_{h(\gamma_1)-B_1} \bar{\omega}\, d\omega \\ &= \operatorname{Re} \frac{1}{2} \int_{\theta_1}^{\theta_2} \left[\bar{\mu}_0 re^{i\theta} + \sum \bar{\mu}_t r^{t+1} e^{i(1-t)\theta} + \sum t\mu_t r^{t+1} e^{i(t-1)\theta} + o(1) \right] d\theta, \end{aligned}$$

where

$$\begin{aligned} \theta_1 &= b_1 r^{-1} + o(r^{-1}), \\ \theta_2 &= (m+1)\pi + \varepsilon b_1 r^{-1} + \varepsilon \operatorname{Im} q^{r-1} + o(r^{-1}), \end{aligned}$$

where $\varepsilon = (-1)^{m+1}$. Now if $s = \pm(1 \pm t) = \pm 2n/(m+1) \neq 0$, $s=1$, or $s=0$, then

$$\text{(6.12)} \qquad \begin{aligned} \int_{\theta_1}^{\theta_2} e^{is\theta}\, d\theta &= \varepsilon r^{-1} \operatorname{Im} q + o(r^{-1}) && \text{if } \delta = \pm 2n/(m+1) \neq 0, \\ &= (1-\varepsilon)i + \varepsilon r^{-1} \operatorname{Im} q + o(r^{-1}) && \text{if } \delta = 1, \\ &= (m+1)\pi + \varepsilon r^{-1} \operatorname{Im} q + o(r^{-1}) && \text{if } \delta = 0. \end{aligned}$$

Hence after integration the two sums in (6.11) are $o(1)$ because $r^t = o(1)$. Only the term involving μ_0 is significant and hence by (6.12) we have

$$\text{(6.13)} \qquad \operatorname{Re} \frac{1}{2i} \int_{h(\gamma_1)-B_1} \bar{\omega}\, d\omega = (1+\varepsilon) r \operatorname{Im} \mu_0/2 + \varepsilon \operatorname{Re} \mu_0 \operatorname{Im} q/2 + o(1).$$

Hence if $m+1$ is odd we are required to impose the condition $\operatorname{Im} \mu_0 = 0$ to keep this term from making $u(r) - l(r)$ diverge. Therefore

$$\text{(6.14)} \qquad \operatorname{Re} \frac{1}{2i} \int_{h(\gamma_1)-B_1} \bar{\omega}\, d\omega = \frac{\varepsilon}{2} \operatorname{Re} \bar{\mu}_0 \operatorname{Im} q + o(1).$$

To estimate the third integral in (6.7) recall that $B_2 \subset V$, a horizontal geodesic. Note that $\omega(B_2(t)) = re^{i\theta_2}(1-t) + \varepsilon(re^{\theta_1} + q)t$, $\omega(h(\gamma_2(t))) = H(\omega(B_2(t))) = \omega(B_2(t)) + \mu_0 + o(1)$. So

$$\text{(6.15)} \qquad \begin{aligned} \operatorname{Re} \frac{1}{2i} \int_{h(\gamma_2)-B_2} \bar{\omega}\, d\omega &= \operatorname{Re} \frac{1}{2i} \int_0^1 \bar{\mu}_0\, d\omega(B_2(t)) \\ &= \operatorname{Re}\, (1/2i) \bar{\mu}_0 (\omega(B_2(1)) - \omega(B_2(0))) \\ &= (\varepsilon/2) \operatorname{Re} q \operatorname{Im} \bar{\mu}_0 + o(1). \end{aligned}$$

Addition of (6.9), (6.14), (6.15) gives an expansion in (6.7). Combining this with (6.5) and (6.6) gives

$$\iint_{J_{h(\gamma)}} |d\omega|^2 - \iint_{J_B} |d\omega|^2 = \tfrac{1}{2}(1+\varepsilon)\,\mathrm{Im}\,(\bar{\mu}_0 q) + o(1).$$

Adding (6.3) and (6.15) gives

$$u(r) = \frac{m+1}{2}\pi r^2 + r\,\mathrm{Im}\,q + \sum_{j=1}^{m-2} B_j(\omega)\,d_j(\omega) + \delta\,\mathrm{Im}\,(\bar{\mu}_0 q), \tag{6.16}$$

where $\delta = (1+\varepsilon)/2$, $\varepsilon = (-1)^{m+1}$.

7. **The lower estimate** $l(r)$. To estimate the $|d\omega|^2$-density of $h(D_r)$ from below we must first change variables. Consider the Q-domains in the global trajectory structure of $d\zeta^2$. From them we remove the finite number of unobstructed trajectories which contain the horizontal geodesics of ∂D or the points in $h^{-1}(Z(d\omega^2))$. (Recall that $Z(d\omega^2)$ is the set of zeros of $d\omega^2$.) We shall continue to call the resulting domains $\{\Delta\}$, end domains $\{E_k\}_{k=1}^{m+1}$, or strip domains $\{S_j\}_{j=1}^{N}$, according to whether $\zeta(E_k) = \{(-1)^{k+1}\,\mathrm{Im}\,\zeta > b_k\}$ or $\zeta(S_j) = \{|\mathrm{Im}\,(\zeta-\zeta_j)| < w_j/2\}$. The relation between these new constants b_k and w_j is the same as before (see Lemma (4.18)). Since the boundaries have $|d\omega|^2$-density zero and the domains are disjoint and dense we may write

$$\iint_{h(D_r)} |d\omega|^2 = \sum \iint_{\Delta_r} |d\omega|^2. \tag{7.1}$$

On Δ we may define ζ conformally; on $h(\Delta)$ ω is defined analytically. We emphasize that these are not necessarily the same ζ on U and ω on $h(U)$ as before. Although $H = \omega \circ h \circ \zeta^{-1}$ on $\zeta(\Delta)$ depends on the choice of the integrals, $|H'(\zeta)|$ does not, because of the nature of the relations (4.2) between the branches. Therefore change variables to obtain

$$\iint_{\Delta_r} |d\omega|^2 = \iint_{\zeta(\Delta_r)} |H'(\zeta)|^2\, d\xi\, d\eta, \tag{7.2}$$

where $\zeta = \xi + i\eta$. Now we use the essence of Schwarz's inequality, $a^2 \geqq 2ab - b^2$ to write

$$\iint_{\zeta(\Delta_r)} |H'(\zeta)|^2\, d\xi\, d\eta \geqq 2 \iint_{\zeta(\Delta_r)} |H'(\zeta)|\, d\xi\, d\eta - \iint_{\zeta(\Delta_r)} d\xi\, d\eta. \tag{7.3}$$

Now the sum of the last integrals is the $|d\zeta|^2$-density of J_r, which may be estimated by Lemma (4.21),

$$\begin{aligned} \sum_{\Delta_r} \iint_{\zeta(\Delta_r)} d\xi\, d\eta &= \iint_{J_r} |d\zeta|^2 \\ &= \frac{m+1}{2}\pi r^2 + r\,\mathrm{Im}\,q + \sum A_j(\zeta) c_j(\zeta) + o(1), \end{aligned} \tag{7.4}$$

where q is the quasi-period and $A_j(\zeta)$, $c_j(\zeta)$ are the constants in the Table of Constants (5.16). Now by Fubini's Theorem

$$\iint_{\zeta(\Delta_r)} |H'(\zeta)|\, d\xi\, d\eta = \int_b^a \left(\int_{l(\eta)} |H'(\zeta)|\, d\xi\right) d\eta. \tag{7.5}$$

But $l(\eta)$ is a $|d\zeta|$ horizontal geodesic and on it $d\xi = |d\zeta|$. Thus

$$\int_{l(\eta)} |H'(\zeta)|\, d\xi = \int_{l(\eta)} |H'(\zeta)|\, |d\zeta| = \int_{h(l(\eta))} |d\omega|. \tag{7.6}$$

Let $g(\eta)$ denote the $|d\omega|$-geodesic which joins the end points of $h(l(\eta))$. Then

$$\int_{h(l(\eta))} |d\omega| \geqq \int_{g(\eta)} |d\omega|, \tag{7.7}$$

with equality if and only if $h(l(\eta)) = g(\eta)$, by the uniqueness of geodesics (3.6). The end points of $g(\eta)$ are values of h taken at points on γ_r. We consider now the original choice of ω, ζ for which $H = \omega \circ h \circ \zeta^{-1}$ has expansion (5.8). By Corollary (5.13) and the expansion (5.8) for H it is clear that h preserves the horizontal directions θ_k. Therefore if the end points of $\zeta(l, \eta)$ are $re^{i\theta(\eta)}$, $re^{i\psi(\eta)}$ we have in an end domain E_k, by Lemma (5.11),

$$\int_{g(\eta)} |d\omega| \geqq (-1)^{k-1} \operatorname{Re} H(re^{i\theta(\eta)}) + (-1)^k \operatorname{Re} H(re^{i\psi(\eta)}), \tag{7.8}$$

in a strip domain S_j, by Lemma (5.11),

$$\int_{g(\eta)} |d\omega| = (-1)^{k+1} \operatorname{Re} H(re^{i\theta(\eta)}) + (-1)^{l+1} \operatorname{Re} H(re^{i\psi(\eta)}) + A_j(\omega) + o(1), \tag{7.9}$$

where H is the standard function on $\zeta(U)$ which has expansion (5.8) (see (5.11)), θ_k, θ_l are the directions of S_j and $A_j(\omega)$ is the ω-hdac corresponding to the directions θ_k, θ_l. Now integration with respect to η on the end domains gives

$$\sum_{k=1}^{m+1} \iint_{\zeta(E_k \cap J_r)} |H'(\zeta)|\, d\xi\, d\eta \geqq \sum_{k=1}^{m+1} \int_{\zeta(E_k \cap \gamma_1)} \operatorname{Re} H(\zeta)\, d \operatorname{Im} \zeta. \tag{7.10}$$

Now in a strip domain we have estimates (7.9) and since the width is independent of r we have, for $c_j(\zeta)$ the width of S_j, $A_j(\omega)$ the hdac of ω with respect to the directions of S_j,

$$\iint_{\zeta(S_j \cap J_r)} |H'(\zeta)|\, d\xi\, d\eta \geqq \int_{\zeta(S_j \cap \gamma_1)} \operatorname{Re} H(\zeta)\, d \operatorname{Im} \zeta + A_j(\omega) c_j(\zeta) + o(1). \tag{7.11}$$

Adding (7.10) and (7.11) we obtain

$$\sum_{\Delta} \iint_{\zeta(\Delta_r)} |H'(\zeta)|\, d\xi\, d\eta \geqq \int_{\zeta(\gamma_1)} \operatorname{Re} H(\zeta)\, d \operatorname{Im} \zeta + \sum_{j=1}^{m-2} A_j(\omega) c_j(\zeta) + o(1). \tag{7.12}$$

Now H has expansion (5.8) and γ_1 has the property $\zeta(\gamma_1(\theta))=re^{i\theta}$, $\theta_1(r)\leqq\theta\leqq\theta_2(r)$. Therefore

$$\begin{aligned}\int_{\gamma_1} &\operatorname{Re} H(\zeta)\, d\operatorname{Im}\zeta \\ &= \tfrac{1}{4}\int_{\theta_1}^{\theta_2}\left(re^{i\theta}+\mu_0+\sum \mu_t r^t e^{it\theta}+re^{-i\theta}+\bar{\mu}_0+\sum \bar{\mu}_t r^t e^{-it\theta}\right) \\ &\qquad \times\left(re^{i\theta}+\sum t\mu_t r^t e^{it\theta}+re^{-i\theta}+\sum t\bar{\mu}_t r^t e^{-it\theta}\right) d\theta+o(1).\end{aligned} \tag{7.13}$$

Now if we multiply these terms, collect in powers of r, and integrate, we see by using (6.12) that (7.13) reduces to

$$\begin{aligned}\int_{\gamma_1} &\operatorname{Re} H(\zeta)\, d\operatorname{Im}\zeta \\ &= \tfrac{1}{4}\int_{\theta_1}^{\theta_2}[(r^2e^{2i\theta}+2r^2+r^2e^{-2i\theta}) \\ &\qquad +(\mu_0+\bar{\mu}_0)(re^{i\theta}+re^{-i\theta})+(\bar{\mu}_{-1}+\mu_{-1})]\, d\theta+o(1) \\ &= ((m+1)/2)\pi r^2+r\operatorname{Im} q+\operatorname{Re}\mu_0\operatorname{Im} q+((m+1)/2)\pi\operatorname{Re}\mu_{-1}+o(1).\end{aligned} \tag{7.14}$$

If we put (7.14) into (7.12) and then put (7.12) and (7.4) into the sum of the terms (7.3) we obtain

$$\begin{aligned}l(r) &= ((m+1)/2)\pi r^2+r\operatorname{Im} q+(m+1)\pi\operatorname{Re}\mu_{-1}+2\operatorname{Re}\mu_0\operatorname{Im} q \\ &\qquad +2\sum_{1}^{m-2} A_j(\omega)c_j(\zeta)-\sum_{j=1}^{m-2} A_j(\zeta)c_j(\zeta)+o(1).\end{aligned} \tag{7.15}$$

8. **Equality statement and other remarks.** (a) Our inequality (5.15) follows by substituting the expansions (6.16), (7.15) for $u(r)$, $l(r)$ into (5.9). Suppose now that equality holds. Then all inequalities must be equalities. In particular (6.1) implies that the complement K of $h(D)$ has zero $|d\omega|^2$-density and since K is bounded from ∞, zero Lebesgue density. In the lower estimates the length-area inequality implies $|H'|\equiv 1$, but the inequalities (7.7) and (7.8) must be equality so $H'(\zeta)\equiv 1$. Therefore $H(\zeta)\equiv\zeta+d$ and $h=\omega^{-1}\circ\zeta$ for appropriate ω on an open subset of D.

Conversely if $h=\omega^{-1}\circ\zeta$ on D then by analytic continuation it must hold at all but a finite number of points in D for appropriate choice of ω. Clearly the lower inequalities must all be equalities. It remains only to show that the Lebesgue area of K is zero. But K is bounded by geodesics of finite length. If int $K\neq\varnothing$ then K would contain a closed geodesic which is ridiculous.

As a corollary we see that $2\sum A_j(\omega)c_j(\zeta)=\sum A_j(\zeta)c_j(\zeta)+\sum B_j(\omega)\, d_j(\omega)$ must also hold when $h=\omega^{-1}\circ\zeta$. Indeed much more must hold. We shall not consider that problem here.

(b) Corollary (4.17) allows an alternate expression for (5.15) which we give here. The upper estimate should be the same $u(r)$. In the lower estimate the estimates on end domains should be the same. We may assume μ_0 is zero since these estimates are independent of the choice of μ_0. On the strip domains $\omega+\mu_0$ sends the curve B_1 into a curve translated by $\pm\operatorname{Re}\mu_0$. Indeed the effect is to translate by $(-1)^{k-1}\operatorname{Re}\mu_0$ in H_k. Therefore if $\theta_{k(j)}$, $\theta_{l(j)}$ are the directions of S_j and $\delta_j=\frac{1}{2}(1+(-1)^{k(j)+l(j)})$ then we have

$$\iint_{\zeta(S_j\cap J_r)} |H'(\zeta)|\,d\xi\,d\eta \geqq 2rc_j(\zeta)+(A_j(\omega)+2\operatorname{Re}\mu_0\delta_j)c_j(\zeta)+o(1)$$

$$= \int_{\zeta(\gamma_1\cap S_j)} \operatorname{Re} H(\zeta)\,d\operatorname{Im}\zeta+A_j(\omega)c_j(\zeta)+o(1).$$

Adding this and comparing results both forms of $l(r)$ yield an interesting relation.

COROLLARY (8.1).

$$2\sum_{j=1}^{m-2}(\operatorname{Re}\mu_0)\delta_j = \operatorname{Re}\mu_0\operatorname{Im}q.$$

We may substitute this into (7.15) to alter the form of (5.15) if we desire.

(c) We have given the formulas necessary to compute μ_{-1}, μ_0 namely the formulas for τ_n, σ_n. We wish to indicate here how to compute the adjustment constants. The strip constants are

$$c_j(\zeta) = \operatorname{Im}\int_{a_j}^{b_j} Q^{1/2}\,dz, \tag{8.2}$$

$$d_j(\omega) = \operatorname{Im}\int_{c_j}^{d_j} P^{1/2}\,dw,$$

where a_j, b_j and c_j, d_j are zeros on the boundaries of those strips S_j, T_j in the global trajectory structures of $d\zeta^2$, $d\omega^2$ respectively. The path is any path joining them in the strip.

The hdac were of the form

$$h_{kj} = \operatorname{Re}[(-1)^k\zeta(z_k)+(-1)^l\zeta(z_l)]+g(z_k, z_l). \tag{8.3}$$

We indicate an obvious method for finding $g(z, w)$ which we call *contraction*. Let γ denote any arc which joins z_k to w_k. Then set

$$F(t, s) = g(z_k, \zeta(s))+\gamma_s$$

where $\gamma_s(t)=\gamma(t)$ for $s\leqq t\leqq 1$. Since it is clear how $g(z_k, \gamma(s))$ changes locally this homotopy gives a canonical way to find $g(z_k, w_k)$. It then remains only to compute its length.

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